

# GENUS TWO ENUMERATIVE INVARIANTS IN DEL-PEZZO SURFACES WITH A FIXED COMPLEX STRUCTURE

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**ABSTRACT.** We obtain a formula for the number of genus two curves with a fixed complex structure of a given degree on a del-Pezzo surface that pass through an appropriate number of generic points of the surface. This is done by extending the symplectic approach of Aleksey Zinger. This enumerative problem is expressed as the difference between the symplectic invariant and an intersection number on the moduli space of rational curves on the surface.

## 1. INTRODUCTION

Enumerative Geometry of rational curves in  $\mathbb{P}_{\mathbb{C}}^2$  is a classical question in algebraic geometry. A natural generalization of it is to ask how many genus  $g$  curves with a fixed complex structure are there of a given degree that pass through the right number of generic points. In [10] and [4] using methods of algebraic and symplectic geometry respectively, Pandharipande and Ionel obtain the following result:

**Theorem 1.1** (R.Pandharipande and E.Ionel). *There is an explicit formula for the number of degree  $d$  genus one curves with a fixed complex structure in  $\mathbb{P}_{\mathbb{C}}^2$  that pass through  $3d - 1$  generic points.*

In [5] and [13] by extending the method of Pandharipande and Ionel respectively, Katz, Qin and Ruan and Zinger obtain the following results:

**Theorem 1.2** (S.Katz, Z.Qin, Y.Ruan and A. Zinger). *There is an explicit formula for the number of degree  $d$  genus two curves with a fixed complex structure in  $\mathbb{P}_{\mathbb{C}}^2$  that pass through  $3d - 2$  generic points.*

In [2], we extended Theorem 1.1 to del-Pezzo surface. Our aim here is to extend Theorem 1.2 for del-Pezzo surfaces.

Let  $X$  be a complex del-Pezzo surface and  $\beta \in H_2(X, \mathbb{Z})$  a given homology class. Let  $n_{g,\beta}^j$  denote the number of genus  $g$  curves with a fixed complex structure that pass through the right number of generic points. For notational convenience, the number  $n_{0,\beta}^j$  will be denoted by  $n_{0,\beta}$ . We prove the following:

**Theorem 1.3.** *Let  $X$  be a complex del-Pezzo surface and  $\beta \in H_2(X, \mathbb{Z})$  a given homology class. Denote*

$$x_i = c_i(TX), \quad \text{and} \quad \delta_{\beta} = \langle x_1, \beta \rangle - 1,$$

*where  $c_i$  denotes the  $i^{\text{th}}$  Chern class. Let  $n_{2,\beta}^j$  denote the number of genus two curves with fixed complex structure  $j$  of degree  $\beta$  in  $X$  that pass through  $\delta_{\beta} - 1$  generic points. If  $X$  is  $\mathbb{P}^2$  blown*

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up  $k$  points, then assume that  $n_{0,\beta-3L} > 0$ , where  $L$  denotes the homology class of a line; if  $X = \mathbb{P}^1 \times \mathbb{P}^1$  then make no assumptions. Then,

$$\begin{aligned} \frac{|\text{Aut}(\Sigma_2^j)|}{2} n_{2,\beta}^j &= n_{0,\beta} \left( (2 + b_2(X))\beta^2 - 10x_2 - x_1^2 + \frac{12x_1^2}{\beta \cdot x_1} \right) \\ &+ \sum_{\substack{\beta_1 + \beta_2 = \beta, \\ \beta_1, \beta_2 \neq 0}} \binom{\delta_\beta - 1}{\delta_{\beta_1}} n_{0,\beta_1} n_{0,\beta_2} (\beta_1 \cdot \beta_2) \left( -\frac{6(\beta_1 \cdot x_1)(\beta_2 \cdot x_1)}{(\beta \cdot x_1)} + \frac{\beta_1^2 \beta_2^2}{2} + 10 \right) \end{aligned} \quad (1.1)$$

where  $|\text{Aut}(\Sigma_2^j)|$  is the order of the group of holomorphic automorphisms of a genus two Riemann surface with fixed complex structure  $j$ ,  $b_2(X)$  denotes the second Betti-number of  $X$  and “ $\cdot$ ” denotes topological intersection.

**Remark 1.4.** We expect that (1.1) is valid even when  $n_{0,\beta-3L} = 0$ . The reason we need to impose the condition  $n_{0,\beta-3L} > 0$  is so that we can use our formula in [1] for  $C_\beta$ , the number of rational cuspidal curves on del-Pezzo surfaces through  $\delta_\beta - 1$  generic points. However, as we explain in the introduction of [1], we expect the formula for  $C_\beta$  to hold even when  $n_{0,\beta-3L} = 0$ . The condition  $n_{0,\beta-3L} > 0$  was imposed to prove a transversality result; that condition is a sufficient condition to prove transversality, it may not be necessary. To use the formula of  $C_\beta$  for  $\mathbb{P}^1 \times \mathbb{P}^1$ , we use the result of Kock [6]. Since there are no assumptions on  $\beta$  for that formula to hold, there are no assumptions on (1.1) when  $X = \mathbb{P}^1 \times \mathbb{P}^1$ . See section 4 for further details.

In [8] and [3] a recursive formula is given to compute the numbers  $n_{0,\beta}$  for del-Pezzo surfaces. Hence, Theorem 1.3 gives  $n_{2,\beta}^j$  explicitly. We have written a C++ program to implement the formula in Theorem 1.3 when  $X$  is  $\mathbb{P}^2$  blown up at  $k \leq 8$  points and compute  $n_{2,\beta}^j$ ; the program is available in the web-page

<https://www.sites.google.com/site/ritwik371/home>

The following two facts are consistency checks for our formula. First of all, define

$$g_\beta := \frac{\beta^2 - x_1 \cdot \beta + 2}{2}.$$

Note that  $g_\beta$  is the genus of a smooth degree  $\beta$  curve. Hence, if  $g_\beta = 0$  or 1 then  $n_{2,\beta}^j$  should be zero. We have verified this assertion for several values of  $\beta$  using our program. For instance, we have verified that if  $X$  is  $\mathbb{P}^2$  blown up at  $k \leq 8$  points and  $\beta$  is any one of the following homology classes,

$$(1, 0), (2, 0), (3, 0), (1, -1), (2, -1), (3, -1), (4, -2, -2) \quad \text{or} \quad (4, -2, -2, -2)$$

then  $n_{2,\beta}^j$  is indeed zero (as expected). Since the hypothesis  $n_{0,\beta-3L} > 0$  doesn't hold in these cases, this consistency check is strictly speaking not valid. However, as we explained in remark 1.4, we do expect our formula to be valid even when  $n_{0,\beta-3L} = 0$ .

Next, it is easy to verify directly that when  $X = \mathbb{P}^1 \times \mathbb{P}^1$ ,  $n_{2,\beta}^j$  is zero if  $\beta$  is of bi-degree  $(a, 0)$ ,  $(a, 1)$  (no restriction on  $a$ ) for some  $a$  or if  $\beta = (2, 2)$ . This is as expected since a curve of bi-degree  $(a, 0)$  is irreducible, the genus of smooth curve of bi-degree  $(a, 1)$  is zero and the genus of a smooth curve of bi-degree  $(2, 2)$  is one; hence  $n_{2,\beta}^j$  should be zero in those cases.

The second consistency check is as follows: it is easy to see by geometric arguments that if  $X$  is  $\mathbb{P}^2$  blown up at  $k \leq 8$  points, then

$$n_{2,dL+\sigma_1 E_1+\dots+\sigma_r E_r}^j = n_{2,dL+\sigma_1 E_1+\dots+\sigma_{r-1} E_{r-1}}^j,$$

if each of the  $\sigma_i$  is  $-1$  or  $0$ . Here  $E_i$  denotes the class of an exceptional divisor. We have verified this assertion in many cases. For instance we have verified that

$$n_{2,4L-E}^j = n_{2,4L+0E}^j = n_{2,4L}^j.$$

The reader is invited to use our program and verify these assertions.

**Remark 1.5.** In [12], Zinger corrects an error in [5] and obtains the same formula as in [13]. The method presented in this paper is an extension of symplectic approach employed in [13]; it would be interesting to see if the algebro-geometric method presented in [5] and [12] can be extended to obtain our formula (1.1).

**Remark 1.6.** In order to recover the formula of Zinger in Theorem 1.1 of [13], we first note that by [11, page 363],

$$n_{0,d} = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2 n_{0,d_1} n_{0,d_2} \left( d_1 d_2 - 2 \frac{(d_1 - d_2)^2}{3d-2} \right). \quad (1.2)$$

Equation (1.1) with  $X = \mathbb{P}^2$  gives us

$$n_{2,d}^j = 3(d^2 - 1)n_{0,d} + n_{0,d} \left( -36 + \frac{36}{d} \right) + \sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} n_{0,d_1} n_{0,d_2} d_1 d_2 \left( -\frac{18d_1 d_2}{d} + \frac{d_1^2 d_2^2}{2} + 10 \right). \quad (1.3)$$

Now substitute the value of  $n_{0,d}$  from equation (1.2) in (1.3) only in the term  $n_{0,d}(-36 + \frac{36}{d})$ ; keep the first term unchanged. That gives us Zinger's formula in Theorem 1.1 of [13], namely

$$n_{2,d}^j = 3(d^2 - 1)n_{0,d} + \frac{1}{2} \sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2 n_{0,d_1} n_{0,d_2} \left( d_1^2 d_2^2 + 28 - 16 \frac{9d_1 d_2 - 1}{3d-2} \right).$$

The reason we set  $|\text{Aut}(\Sigma_2^j)| = 2$  is because Zinger states his formula for a generic  $j$ . However, his arguments go through for any  $j$ ; one simply divides out the difference between the symplectic invariant and the correction term by a different factor (see remark 2.1).

Recently, the problem of enumerating genus  $g$  curves with a fixed complex structure has also been studied by tropical geometers. In [7], Kerber and Markwig compute the number of tropical elliptic curves in  $\mathbb{P}^2$  with a fixed  $j$ -invariant. Combined with the correspondence theorem proved by Len and Ranganathan in [9], one can conclude that the number computed is indeed the same as the number of plane elliptic curves with a fixed  $j$ -invariant. That gives a tropical proof of Theorem 1.1. In [9], Len and Ranganathan also obtain a formula for the number of elliptic curves with a fixed  $j$ -invariant of a given degree for Hirzebruch surfaces, using methods from tropical geometry.

The problem of counting genus two curves with a fixed complex structure is also currently being investigated by tropical geometers. It would be interesting if Zinger's formula (and more generally, the formula obtained here) can be recovered by tropical methods.

## 2. ENUMERATIVE VERSUS SYMPLECTIC INVARIANT

Let us now explain the basic idea to compute  $n_{2,\beta}^j$ . Let  $(X, \omega)$  be a compact semipositive symplectic manifold of dimension  $2m$  and  $\beta \in H_2(X, \mathbb{Z})$  a homology class. Let  $k$  be a nonnegative integer such that  $k + 2g \geq 3$ . Let  $[\alpha_1], \dots, [\alpha_k]$  and  $[\gamma_1], \dots, [\gamma_l]$  be integral homology classes in  $H_*(X, \mathbb{Z})$  such that

$$\sum_{i=1}^k 2m - \deg(\alpha_i) + \sum_{j=1}^l (2m - 2 - \deg(\gamma_j)) = 2m(1 - g) + 2\langle c_1(TX), \beta \rangle. \quad (2.1)$$

Fix cycles  $A_i$ ,  $1 \leq i \leq k$ , and  $B_j$ ,  $1 \leq j \leq l$ , on  $M$  representing the cohomology classes  $\alpha_i$  and  $\gamma_j$ . Fix a compact Riemann surface  $\Sigma_g$  of genus  $g$ ; its complex structure will be denoted by  $j$ . Define

$$\mathcal{M}_{g,k}^{\nu,j}(X, \beta; \alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) := \{(u, y_1, \dots, y_k) \in \mathcal{C}^\infty(\Sigma_g, X) \times X^k \mid \\ \bar{\partial}_{j,J} u = \nu, u(y_i) \in A_i \forall i = 1, \dots, k, \text{Im}(u) \cap B_j \neq \emptyset \forall j = 1, \dots, l\},$$

where  $\nu : \Sigma_g \times X \rightarrow T^*\Sigma_g \otimes TX$  is a generic smooth perturbation. The symplectic invariant (or the Ruan–Tian invariant) is defined to be the signed cardinality of the above set, i.e.,

$$\text{RT}_{g,\beta}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) := \pm |\mathcal{M}_{g,k}^{\nu,j}(X, \beta; \alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l)|.$$

When  $k = 0$ , we denote the invariant as

$$\text{RT}_{g,\beta}(\emptyset; \gamma_1, \dots, \gamma_l).$$

Similarly, when  $l = 0$  we denote the invariant as

$$\text{RT}_{g,\beta}(\alpha_1, \dots, \alpha_k; \emptyset).$$

If (2.1) is not satisfied, then we formally define the invariant to be zero.

A natural question to ask is whether the symplectic invariant coincides with the enumerative invariant  $n_{g,\beta}^j$ . It is stated in [11, page 267] [11] that when  $g = 0$  and  $X$  is a del-Pezzo surface, the symplectic invariant is same as the enumerative invariant, meaning,

$$\text{RT}_{0,\beta}(\emptyset; p_1, \dots, p_{\delta_\beta-1}) = n_{0,\beta}.$$

However, starting from  $g = 1$ , the enumerative invariant is no longer the same as the symplectic invariant. Before we explain why that is so, let us recapitulate a few definitions.

Consider the moduli space of rational degree  $\beta$  curves on  $X$  that represent the class  $\beta \in H_2(X, \mathbb{Z})$  and are equipped with  $n$  ordered marked points. Let  $\mathcal{M}_{0,n}(X, \beta)$  denote the equivalence classes of such curves. In other words,

$$\mathcal{M}_{0,n}(X, \beta) := \{(u, y_1, \dots, y_n) \in \mathcal{C}^\infty(\mathbb{P}^1, X) \times (\mathbb{P}^1)^n \mid \bar{\partial}u = 0, u_*[\mathbb{P}^1] = \beta\} / \text{PSL}(2, \mathbb{C}),$$

with  $\text{PSL}(2, \mathbb{C})$  acting diagonally on  $\mathbb{P}^1 \times (\mathbb{P}^1)^n$ . For any  $k \leq n$ , let

$$\mathcal{M}_{0,n}(X, \beta; p_1, \dots, p_k) \subset \mathcal{M}_{0,n}(X, \beta)$$

be the subspace consisting of rational curves with  $n$  marked points such that the  $i$ -th marked point is  $p_i$  for all  $1 \leq i \leq k$ , so,

$$\mathcal{M}_{0,n}(X, \beta; p_1, \dots, p_k) := \{[u, y_1, \dots, y_n] \in \mathcal{M}_{0,n}(X, \beta) \mid u(y_i) = p_i \forall i = 1, \dots, k\}.$$

Let  $\overline{\mathcal{M}}_{0,n}(X, \beta)$  denote the stable map compactification of  $\mathcal{M}_{0,n}(X, \beta)$ .

We now return to the discussion of why the symplectic invariant is not equal to the enumerative invariant. In general, the following statement is true

$$\text{RT}_{g,\beta} = |\text{Aut}(\Sigma_g^j)| n_{g,\beta}^j + \text{CR}_{g,\beta}, \quad (2.2)$$

where  $\text{CR}_{g,\beta}$  denotes a correction term. Let us explain what this term means and why it arises. First we note that the factor of  $|\text{Aut}(\Sigma_g^j)|$  is there because in the definition of the symplectic invariant it is not modulo the automorphisms. Hence, if  $u : (\Sigma_g, j) \rightarrow X$  is a solution to the  $\bar{\partial}$ -equation, then there will be  $|\text{Aut}(\Sigma_g^j)|$  new solutions close to  $u$  to the perturbed  $\bar{\partial}$ -equation. Next, we note that when  $g = 2$ , as  $\nu \rightarrow 0$ , a sequence of  $(J, \nu)$ -holomorphic maps can also converge to a bubble tree whose base (the torus) is a constant (ghost) map [13, page 5]. These maps will also contribute to the computation of  $\text{RT}_{2,\beta}$  invariant. This extra contribution is defined to be the correction term  $\text{CR}_{2,\beta}$ . This correction term  $\text{CR}_{2,\beta}$  is computed in [13] when  $X = \mathbb{P}^2$  by expressing it as the intersection of certain tautological classes on the moduli space of rational curves and in terms of the number of rational cuspidal curves (through the

right number of generic points). However, the gluing constructions in [15] and [13] are valid in general for Kähler manifolds [15, page 8]. Hence, the formula obtained in [13] for the genus two correction term expressed using the characteristic number of cuspidal curves and the intersection of tautological classes is valid even when  $X$  is a del-Pezzo surface. In [1] the characteristic number of rational cuspidal curves and the intersection of these tautological classes for del-Pezzo surfaces is computed. Hence, combining the result of [13] and [1] it is possible to compute the correction term for any del-Pezzo surface. To compute the symplectic invariant we use the formula presented in [11]. Hence, using (2.2) we can compute the enumerative number  $n_{2,\beta}^j$ .

**Remark 2.1.** The symplectic invariant  $\text{RT}_{g,\beta}$  does not depend on  $j$ . Neither does the correction term  $\text{CR}_{g,\beta}$ . It is only the enumerative invariant  $n_{g,\beta}^j$  that depends on  $j$ . Equation (2.2) implies that

$$n_{g,\beta}^j = \frac{\text{RT}_{g,\beta} - \text{CR}_{g,\beta}}{|\text{Aut}(\Sigma_g^j)|}.$$

It is because of the presence of the factor  $|\text{Aut}(\Sigma_g^j)|$ , that  $n_{g,\beta}^j$  depends on  $j$ . If  $g = 2$  and  $j$  is a generic complex structure on a genus two surface, then  $|\text{Aut}(\Sigma_g^j)| = 2$ .

### 3. COMPUTATION OF THE SYMPLECTIC INVARIANT

We compute the symplectic invariant  $\text{RT}_{2,\beta}$  using formulas (1.1) and (1.2) in [11, page 263]. Throughout the discussion  $X$  is a complex del-Pezzo surface (either  $\mathbb{P}^2$  blown up at  $k \leq 8$  points or  $\mathbb{P}^1 \times \mathbb{P}^1$ ). Let  $e_0, e_1, e_2, \dots, e_n, e_{n+1}$  be a basis for  $H_*(X, \mathbb{Z})$ . Let  $e_0$  denote the 0-th homology class of a point and  $e_{n+1}$  the 4-th homology class defined by the whole space  $[X]$ . The remaining  $e_1, \dots, e_n$  are elements of  $H_2(X, \mathbb{Z})$ . Occasionally  $e_0$  and  $e_{n+1}$  will be denoted by  $\text{pt}$  and  $X$  respectively. For notational conveniences, we will also use

$$f_1 := e_1, f_2 := e_2, \dots, f_n := e_n.$$

The reason for this duplication of notation will be clear soon. Note that there is no such thing as  $f_0$  or  $f_{n+1}$ . Hence, if we encounter a term  $f_i$ , it is understood that  $i$  is between 1 and  $n$ .

Define

$$g_{ij} := e_i \cdot e_j \quad \text{and} \quad g^{ij} := (g^{-1})_{ij}.$$

If the degree of  $e_i$  and  $e_j$  do not add up to the dimension of  $X$ , then define  $g_{ij} := 0$ . Also, define

$$[p] := p_1, p_2, \dots, p_{\delta_\beta - 1}.$$

We will also be using the Einstein summation convention, since it will make the subsequent computation much easier to read.

To compute the symplectic invariant, using [11, page 263, (1.2)], it follows that

$$\begin{aligned} \text{RT}_{2,\beta}(\emptyset; [p]) &= \text{RT}_{1,\beta}(e_i, e_j; [p])g^{ij} \\ &= \text{RT}_{0,\beta}(e_i, e_j, e_k, e_l; [p])g^{ij}g^{kl} \\ &= \text{RT}_{0,\beta}(\text{pt}, X, e_k, e_l; [p])g^{kl} + \text{RT}_{0,\beta}(X, \text{pt}, e_k, e_l; [p])g^{kl} \\ &\quad + \text{RT}_{0,\beta}(e_i, e_j, \text{pt}, X; [p])g^{ij} + \text{RT}_{0,\beta}(e_i, e_j, X, \text{pt}; [p])g^{ij} \\ &\quad + \text{RT}_{0,\beta}(f_i, f_j, f_k, f_l; [p])g^{ij}g^{kl} \\ &= 4n_{0,\beta}(\beta \cdot f_i)(\beta \cdot f_j)g^{ij} + \text{RT}_{0,\beta}(f_i, f_j, f_k, f_l; [p])g^{ij}g^{kl}. \end{aligned} \tag{3.1}$$

Next, using [11, page 263, (1.1)] it follows that

$$\text{RT}_{0,\beta}(f_i, f_j, f_k, f_l; [p])g^{ij}g^{kl}$$

$$\begin{aligned}
&= \text{RT}_{0,\beta}(f_i, f_j, \text{pt}; [p]) \text{RT}_{0,0}(X, f_k, f_l; \emptyset) g^{ij} g^{kl} + \text{RT}_{0,0}(f_i, f_j, X; \emptyset) \text{RT}_{0,\beta}(f_k, f_l, \text{pt}; [p]) g^{ij} g^{kl} \\
&+ \sum_{\beta_1 + \beta_2 = \beta} \binom{\delta_\beta - 1}{\delta_{\beta_1}} n_{0,\beta_1} n_{0,\beta_2} (\beta_1 \cdot f_i) (\beta_1 \cdot f_j) (\beta_1 \cdot f_m) (\beta_2 \cdot f_n) (\beta_2 \cdot f_k) (\beta_2 \cdot f_l) g^{ij} g^{kl} g^{mn}. \quad (3.2)
\end{aligned}$$

Note that

$$\text{RT}_{0,0}(X, f_k, f_l; \emptyset) g^{kl} = (f_k \cdot f_l) g^{kl} = b_2(X), \quad (3.3)$$

where  $b_2(X)$  is the dimension of  $H_2(X, \mathbb{Z})$ .

Next, we observe that

$$(\gamma_1 \cdot f_i) (\gamma_2 \cdot f_j) g^{ij} = \gamma_1 \cdot \gamma_2 \quad \forall \gamma_1, \gamma_2 \in H_2(X, \mathbb{Z}). \quad (3.4)$$

Using equation (3.1), (3.2), (3.3) and (3.4) it follows that

$$\text{RT}_{2,\beta}(\emptyset; [p]) = (4 + 2b_2(X)) n_{0,\beta} \beta \cdot \beta + \sum_{\beta_1 + \beta_2 = \beta} \binom{\delta_\beta - 1}{\delta_{\beta_1}} \beta_1^2 \beta_2^2 (\beta_1 \cdot \beta_2) n_{0,\beta_1} n_{0,\beta_2}. \quad (3.5)$$

#### 4. COMPUTATION OF THE CORRECTION TERM

We now compute the correction term to the symplectic invariant. Let

$$\mathcal{L}_i \longrightarrow \overline{\mathcal{M}}_{0,n}(X, \beta) \quad \text{and} \quad \text{ev}_i : \overline{\mathcal{M}}_{0,n}(X, \beta) \longrightarrow X$$

be the universal tangent line bundle and the evaluation map respectively at the  $i$ -th marked point. Let  $C_\beta$  denote the number of rational curves in  $X$  passing through  $\delta_\beta - 1$  generic points that have a cusp. For notational convenience  $\mathcal{M}_{0,\delta_\beta}(X, \beta; p_1, \dots, p_{\delta_\beta-1})$  will also be denoted by  $\mathcal{M}$ . We recall from [1], that

$$\begin{aligned}
\langle c_1(\mathcal{L}_{\delta_\beta}^*) \text{ev}_{\delta_\beta}^*(x_1), [\overline{\mathcal{M}}] \rangle &= -\frac{(x_1 \cdot x_1)}{(\beta \cdot x_1)} n_{0,\beta} \\
&+ \frac{1}{2(\beta \cdot x_1)} \sum_{\substack{\beta_1 + \beta_2 = \beta, \\ \beta_1, \beta_2 \neq 0}} \binom{\delta_\beta - 1}{\delta_{\beta_1}} n_{0,\beta_1} n_{0,\beta_2} (\beta_1 \cdot \beta_2) (\beta_1 \cdot x_1) (\beta_2 \cdot x_1), \quad (4.1)
\end{aligned}$$

$$C_\beta = \left( x_2([X]) - \frac{x_1 \cdot x_1}{\beta \cdot x_1} \right) n_{0,\beta} + \sum_{\substack{\beta_1 + \beta_2 = \beta, \\ \beta_1, \beta_2 \neq 0}} \binom{\delta_\beta - 1}{\delta_{\beta_1}} n_{0,\beta_1} n_{0,\beta_2} (\beta_1 \cdot \beta_2) \left( \frac{(\beta_1 \cdot x_1)(\beta_2 \cdot x_1)}{2(\beta \cdot x_1)} - 1 \right). \quad (4.2)$$

Note that when  $X$  is  $\mathbb{P}^2$  blown up at  $k \leq 8$  points, equation (4.2) holds by [1], provided  $n_{0,\beta-3L} > 0$ . That is the reason we impose that condition in Theorem 1.3. When  $X = \mathbb{P}^1 \times \mathbb{P}^1$ , equation 4.2 is also valid by the result of Kock [6]. Equation (4.1) is valid in both the cases; the proof presented in [1] for (4.1) is valid irrespective of whether  $X$  is  $\mathbb{P}^2$  blown up at  $k \leq 8$  points or  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Define  $|\mathcal{V}_2|$  to be the total number of two component rational degree  $\beta$  passing through  $\delta_\beta - 1$  generic points and keeping track of the point at which the two components intersect. Hence

$$|\mathcal{V}_2| = \frac{1}{2} \sum_{\substack{\beta_1 + \beta_2 = \beta, \\ \beta_1, \beta_2 \neq 0}} \binom{\delta_\beta - 1}{\delta_{\beta_1}} n_{0,\beta_1} n_{0,\beta_2} (\beta_1 \cdot \beta_2).$$

The factor of  $\beta_1 \cdot \beta_2$  is there since we are keeping track of the point of intersection of the  $\beta_1$  curve and the  $\beta_2$  curve. We are now ready to compute the correction term by using the result of Zinger for  $\mathbb{P}^2$ . But before that, let us make a small digression. The mentioned result of Zinger, which we will be using is part of his PhD thesis [14] and has been published in [13]. However,

we will be referring to the authors PhD thesis [14] as opposed to the paper [13], because the specific result which we want to make use of is explained in a better way in the thesis. We feel it will be easier for the reader to refer to Zinger's thesis as opposed to the published paper [13] to see how to modify the arguments in the case of del-Pezzo surfaces. The PhD thesis is available at

<http://dspace.mit.edu/handle/1721.1/8402?show=full>

It is shown in [14] that the correction term  $\text{CR}_{2,\beta}$  is a sum of four terms. More precisely, equation 9.20 in [14] states that

$$\text{CR}_{2,\beta} = n_1^1 + 2n_1^2 + 18n_1^3 + n_2^1, \quad (4.3)$$

where we will shortly explain how to compute these four numbers  $n_1^1, n_1^2, n_1^3$  and  $n_2^1$ . Note that we are very closely following the discussion in Zinger's thesis [14] and hence we are using his notation. The numbers  $n_1^1, n_1^2, n_1^3$  and  $n_2^1$  are not to be confused with  $n_{g,\beta}^j$ .

By Lemma 9.1, Corollary 9.3 and Lemma 9.4 in [14], we conclude that

$$n_1^2 = 2C_\beta, \quad n_1^3 = C_\beta \quad \text{and} \quad n_2^1 = 4|\mathcal{V}_2|. \quad (4.4)$$

We now explain how to compute  $n_1^1$ ; this is in fact the reason why we are referring to the thesis of Zinger [14] as opposed to the paper [13], because the argument in the former is much easier to follow and adapt to the case of del-Pezzo surfaces. Using Lemma 9.5 in [14], we conclude that

$$n_1^1 = 2\langle c_1(\mathcal{L}_{\delta_\beta}^*) \text{ev}_{\delta_\beta}^*(x_1), [\overline{\mathcal{M}}] \rangle + (2x_1^2 - 2x_2)n_{0,\beta}. \quad (4.5)$$

To see why this is so, we simply unravel the computation of  $N(\alpha_1)$  in equation 9.1 in [14]. In order to do that we use [14, Corollary 5.17]. In [14, Corollary 5.17], set

$$\begin{aligned} L_\Sigma &:= T\Sigma \longrightarrow \Sigma, & V_\Sigma &:= \mathcal{H}^{0,1} \times \Sigma \longrightarrow \Sigma, \\ L_{\mathcal{M}} &:= \mathcal{L}_{\delta_\beta} \longrightarrow \overline{\mathcal{M}}, & \text{and} & \quad V_{\mathcal{M}} := \text{ev}_{\delta_\beta}^* TX \longrightarrow \overline{\mathcal{M}}. \end{aligned}$$

Note that  $V_\Sigma := \mathcal{H}^{0,1} \times \Sigma \longrightarrow \Sigma$  is a trivial bundle, hence  $c_1(V_\Sigma) = 0$ . Now, unraveling the computation of  $N(\alpha_1)$  in equation 9.1 in [14] and using [14, Corollary 5.17], we immediately get (4.5). Using (4.4), (4.5) and (4.3) we obtain the formula for  $\text{CR}_{2,\beta}$ .

Finally, using the formula for the symplectic invariant (3.5) and (2.2) we obtain the formula in Theorem 1.3.

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